Economic Growth

The Ramsey-Cass-Koopmans Model

Preface

- This slide set is part of my lecture "Economic Growth" where I present the RCK model in Chapter V. In case of questions, comments and/or suggestions write an email to sacht[at]economics.uni-kiel.de.

- Using Matlab, the script 'RCK_Local_Dynamics.m' ['RCK_Productivity_Shock.m'] can be executed in order to reproduce the Figures to be found on the slides 40, 42 and 44 [46 and 49].

- Note that the following denotation of parameters applies. $E(t)$ is labour efficiency at time 't' (under the concept of Harrod-neutrality) where its growth rate is given by 'x'. 'a', 'δ',and 'α' denote productivity, the depreciation rate of capital and the income share of capital, respectively. The nominal wage is given by 'ω'.

- For the levels of capital, labour and output/income we consider K, L and Y. Levels and growth rates in per capita terms are represented by lowercase letters, e.g. y and $\widehat{y}$, respectively.

Household's Optimization Problem

- Assume identical households, each supplying inelastically one unit of labour. A household represents the founder of a dynasty lasting forever.

- Its utility depends on consumption $c(t)$ at all times, from now on, irrespective of whether it is consumption of the present generation or that of a descendent.

- *Future* utility flows get a *lower weight* in the present utility level U the *further away* in the future they will be enjoyed.

- We call u the current utility (a flow, measured in "utils" per year), and U the present utility (a stock, measured in "utils").

- $u(\cdot)$ is a strictly concave function measuring the current utility flow at t as a function of the current consumption flow $c(t)$. We assume $n = 0$ for the sake of simplicity.

- $\rho > 0$ is the subjective discount **rate**, a preference parameter.

- It must not be confused with the discount **factor** $D(t)$. The latter denotes the factor by which a future cash flow (due to asset holding) must be multiplied in order to obtain the corresponding present value.

- $D(t)$ is not a parameter but an endogenous variable since it depends on the (average) market interest rate (such that $\hat{D}(t) = -\iota(t)$ holds).

- Hence, the utility function becomes:

$$U = \int_0^\infty u\Big(c(t)\Big) \exp(-\rho t) dt. \tag{1}$$

- The household hold an *asset* $\tilde{a}(t)$ in form of a loan. It is measured in real terms, i.e., in units of consumables. For the amount (value) of the asset $\tilde{a}(t) \gtreqless 0$ holds. Therefore, $\tilde{a}(t) < 0$ implies that the household is in debt.

- In the capital market equilibrium $\tilde{a}(t)L(t) = K(t)$ applies. It it assumed that $\tilde{a}$ and K are perfect substitutes. Hence, they must pay the same rate of return given by the market interest rate $\iota(t)$.

- The household's asset $\tilde{a}(t)$ develops according to

$$\dot{\tilde{a}}(t) = w(t) + \iota(t)\tilde{a}(t) - c(t). \tag{2}$$

- The utility function (1) is maximized subjected to the budget constraint (2). It follows that optimality of the household's decision has two implications (necessary *optimality conditions*).

Household's First-Order Conditions

- First, for all t, one extra Euro available at t ("later") must contribute to U marginally the same as $D(t)$ Euros (the present value of the extra Euro) available at $t = 0$ ("now").

- Otherwise, the household could raise present utility by either borrowing (if $D(t)$ Euros now contribute more then one Euro later) or lending (if one Euro later contributes more than $D(t)$ Euros now).

- Formally, as we apply the *Hamiltonian* optimization technique, we arrive at

  $$u'\Big(c(t)\Big)\,\mathrm{exp}(-\rho t) = D(t)u'\Big(c(0)\Big)$$

  or, translated into growth rates, at

  $$\varepsilon_{u':c}\,\widehat{c} - \rho = -\iota.$$

- This can be rewritten using the elasticity of marginal utility with respect to consumption, $\varepsilon_{u':c} < 0$. Defining $\theta := -\varepsilon_{u':c} > 0$, the condition becomes

$$\widehat{c} = (1/\theta)(\iota - \rho). \tag{3}$$

- This is the celebrated **Keynes-Ramsey rule (KRR)**, the *first* implication of **optimality**.

- $(1/\theta)$ is called the intertemporal elasticity of substitution.

- In general, it is a function of c, but often one takes the special case where it is constant. A special case of this special case is $u(c) = \log c$, thus $u'(c) = 1/c$ thus $\theta = 1$.

- A household chooses $c(t)$ for all times under *perfect foresight* and with a perfect capital market in a way, such that U is maximal under the "no chain letter" (NCL; where "CL" is also known as "Ponzi Game") constraint,

$$\lim_{t\to\infty} D(t)\tilde{a}(t) \geq 0. \tag{4}$$

- The NCL implies that in the limit the value of the asset $\tilde{a}(t)$ is *at least* zero in discounted terms; i.e. in period $t = 0$.

- As we assume the inequality (4) to be violated,

$$\lim_{t\to\infty} D(t)\tilde{a}(t) < 0$$

  holds.

- This implies that the household will *borrow today* in order to finance *consumption today*. In the following time periods the household is going to *finance the outstanding interest payments* (on the assets hold from the previous period) by *borrow new assets* and so on.

- This is the "chain letter". As a result the level of debt (i.e. the negative value of the asset $\tilde{a}(t)$) will grow forever and, hence, will not paid at the infinite time horizon $T = \infty$.

- The violation of (4) then implies that the household will consume in $t = 0$ "for free" since there is debt at $T = \infty$.

- However, there is no free lunch! If debt is possible at $T = \infty$ it must be mimicked by a surplus of "another" household:

$$\lim_{t\to\infty} D(t)\tilde{a}(t) > 0.$$

This is indeed implausible since a surplus at $T = \infty$ would imply a lower level of consumption (and, hence, utility) for this specific household while holding assets instead.

- This observation leads directly to the *second* implication of **optimality**, which is the so-called **transversality condition (TVC)**,

$$\lim_{t\to\infty} D(t)\tilde{a}(t) = 0. \tag{5}$$

- We have shown that the two conditions, KRR and TVC, are necessary. They are also sufficient due to concavity of the objective.

Market Equilibrium

- Production is as in the Solow model with an exogenous technological progress.

- Equilibrium on the output market requires

$$\dot{K} + \delta K + Lc = F(K, EL), \tag{6}$$

(gross investment *plus* consumption equals output).

- Capital market equilibrium requires $\tilde{a}L = K$ and $\iota = f'(k) - \delta$. Finally, labour market equilibrium requires $E\Big(f(k) - kf'(k)\Big) = w$. Therefore consider the optimality conditions obtained from firm's profit maximization.

- Dividing (6) by K gives

$$\widehat{K} = F(K, LE)/K - Lc/K - \delta = f(k)/k - g/k - \delta,$$

with $g := c/E$ (consumption *per effective worker*; warning: this is denoted $\widehat{c}$ in [2, Chapter 2.3]; don't confuse with our growth rate notation!).

- As, by assumption, $\widehat{L} = 0$ and, by definition, $k = K/(EL)$, one has $\widehat{k} = \widehat{K} - x$, and therefore

$$\widehat{k} = f(k)/k - g/k - (\delta + x). \tag{7}$$

- The definition of g implies $\widehat{g} = \widehat{c} - x$. By the KRR (3) we thus get

$$\widehat{g} = (1/\theta)(\iota - \rho) - x.$$

- Using $\iota = f'(k) - \delta$ we thus finally obtain

$$\widehat{g} = (1/\theta)(f'(k) - \delta - \rho) - x. \tag{8}$$

- Equations (7) and (8) are two dynamic equations describing **simultaneously** the movement of k and g (in growth rates) *over time*.

- For k the starting value is given by the history of the world until $t = 0$, $k(0) = K_0/L$. E is normalized to $E(0) = 1$. Therefore, the capital stock is a **predetermined** variable.

- For g there is no such given starting value $g(0)$. It can *jump* freely to any value compatible with the equilibrium conditions that have to hold from now on forever. Therefore, consumption is a **jump** variable.

- The degree of freedom is closed by the TVC requiring (using $\tilde{a} = K/L = EK/(EL) = Ek$)

$$\lim_{t\to\infty} D(t)\tilde{a}(t) = \lim_{t\to\infty} D(t)E(t)k(t) = 0. \tag{9}$$

- It can be shown that the market equilibrium is *Pareto optimal*. By optimality it is meant that *no reallocations* occur in equilibrium. See [1], Chapter 8.3 and [2], Chapter 2.4 for more details.

- Not that the saving rate s is (now) **endogenously** determined in the RCK model.

- From the previous chapters we know (with g instead of c):

$$g = (1-s)y = (1-s)f(k) = f(k) - sf(k).$$

- As we solve this equation for s and translate it into growth rates we have:

$$\underbrace{\frac{\text{d } \log(s)}{\text{d}t}}_{=\hat{s}} = \underbrace{\frac{\text{d } \log(1)}{\text{d}t}}_{=0} - \underbrace{\frac{\text{d } \log(g)}{\text{d}t}}_{=\hat{g}} + \underbrace{\frac{\text{d } \log[f(k)]}{\text{d}t}}_{=\hat{y}}.$$

- As we consider the CD production function for simplicity, we obtain for the last term:

$$\hat{y} = \frac{\text{d } \log[f(k)]}{\text{d}k} \cdot \frac{\text{d}k}{\text{d}t} = \frac{\text{d}[\log(a) + \alpha\log(k)]}{\text{d}k} \cdot \dot{k} = \alpha/k \cdot \dot{k} = \alpha\hat{k}.$$

- Finally, the growth rate of the saving rate is then given by:

$$\hat{s} = \hat{y} - \hat{g} = \alpha\hat{k} - \hat{g}. \qquad (10)$$

Steady State & Isoclines

- We plot the $(\hat{k} = 0)$-isocline (k-isocline, for short) and the $(\hat{g} = 0)$-isocline (g-isocline, for short) in (k, g) phase space, with k on the horizontal axis.

- The **k-isocline** shows all (k, g) combinations at which $\hat{k} = 0$. According to (7) it is given by

$$g = f(k) - (\delta + x)k. \tag{11}$$

- This is a strictly concave curve, originating at $(0, 0)$ with infinite slope, due to Inada. For k large enough, the slope must become negative, again due to Inada.

- Therefore there is one and only one k at which the curve attains its maximum, the "golden" k_g with

$$f'(k_g) = \delta + x. \tag{12}$$

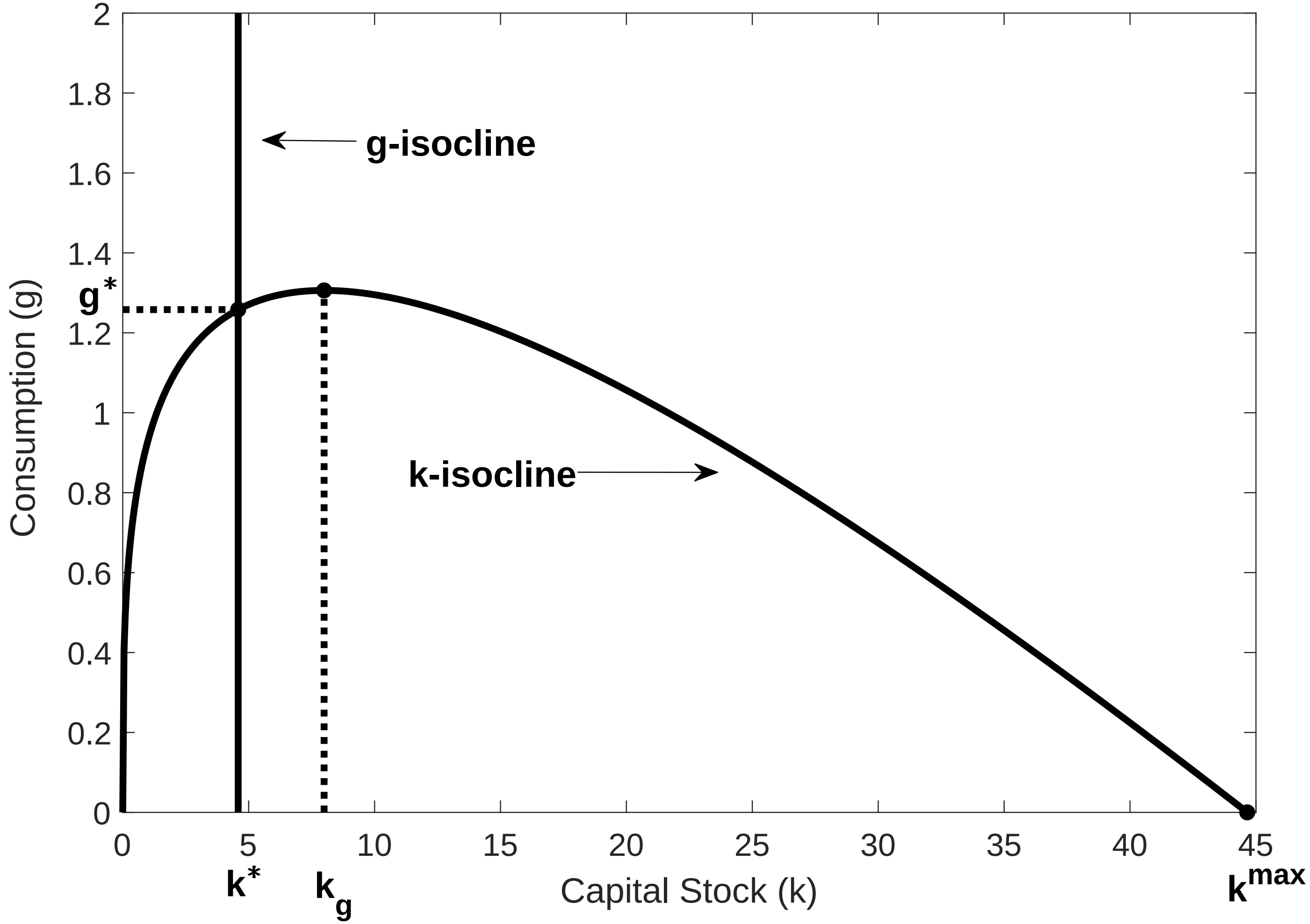
- The **g-isocline** shows all (k, g) combinations at which $\widehat{g} = 0$. According to (8) this is a vertical line at k^* with

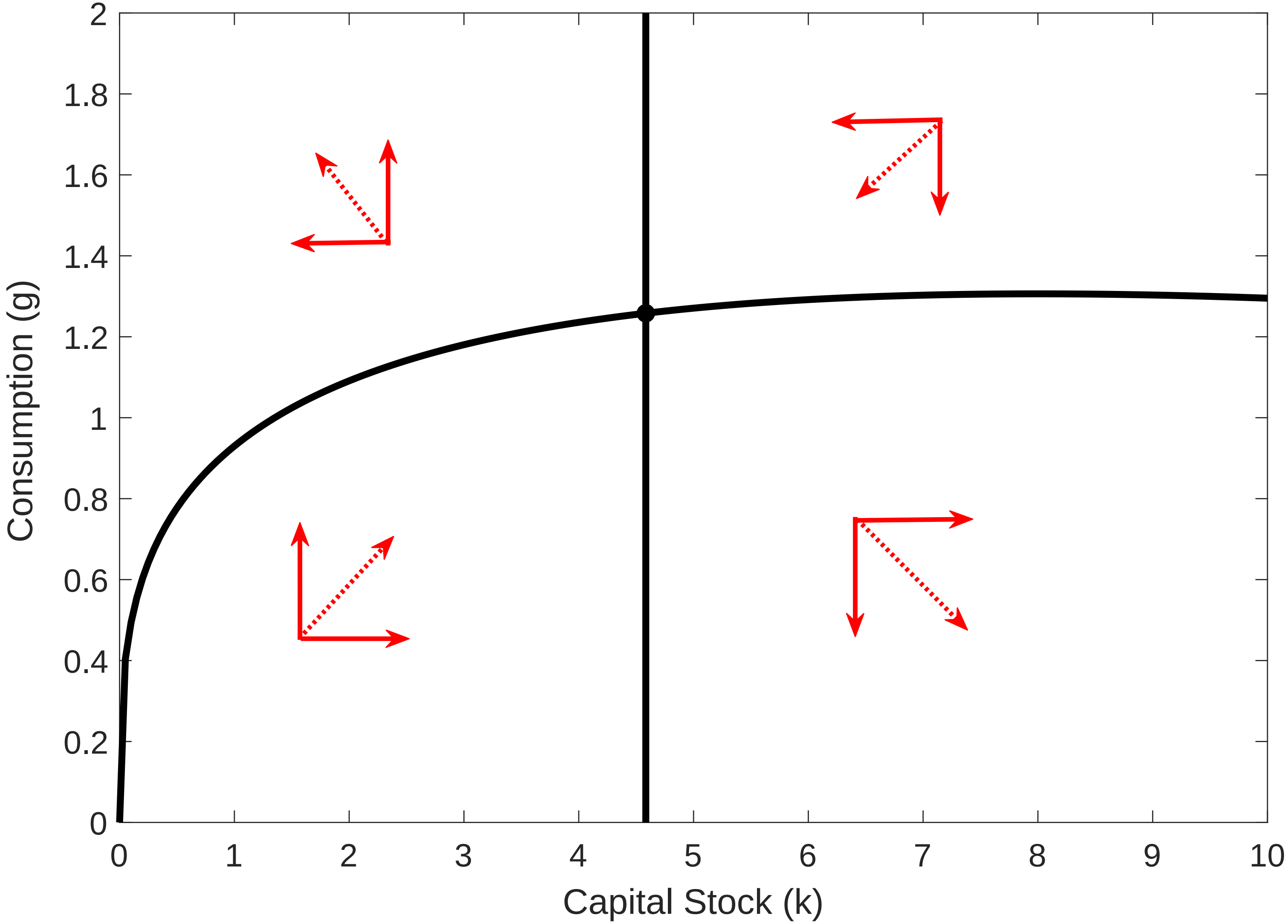
$$f'(k^*) = \delta + \rho + \theta x. \tag{13}$$

- The steady state is where both isoclines cut because here $\widehat{k} = \widehat{g} = 0$ holds. Here, the optimal steady state equilibrium is denoted by (k^*, g^*).

- As in the Solow model with an exogenous technological progress, K and Y (under the assumption $n = 0$) as well as y (output *per capita*) together with w all grow at the rate x (while the interest rate and the income distribution remain constant).

Graphical Solution: the Phase Diagram




Consumption (g)
Capital Stock (k)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
0
1
2
3
4
5
6
7
8
9
10

Analytical Solution: Local Dynamics

Linearization around the Steady State

- Note that we consider a **non-linear** homogeneous system of ODEs:

$$\dot{k} = f(k) - g - (\delta + x)k, \quad (14)$$
$$\dot{g} = (1/\theta)[f'(k) - \delta - \rho - \theta x]g \quad (15)$$

  (cf. equations (7) and (8)).

- Our goal is to determine the time paths for k and g. Therefore, we are going to *linearize* the system *around the steady state*.

- A local analysis is only valid as the system dynamics (after a shock) do not originate "far" away from the optimal steady state equilibrium.

- We limit our (algebraic) analysis to a locally instead of globally stable outcome. In order to deal with global dynamics, more advanced solution techniques must be applied.

- They are rather complicated and the discussion of them would go beyond the scope of this course. See [3], Chapter 20 onwards for more details.

- We linearize the system by applying the *Taylor expansion* of first order. In general, the following formula is considered:

$$\dot{z}_i = \dot{z}_i^* + \frac{\partial f^i(z_1, z_2)}{\partial z_1}\Big|_{\text{SS}} \cdot (z_1 - z_1^*) + \frac{\partial f^i(z_1, z_2)}{\partial z_2}\Big|_{\text{SS}} \cdot (z_2 - z_2^*)$$

for $i = \{1, 2\}$. The abbreviation 'SS' denotes steady state, i.e. the partial derivatives are evaluated at the SS.

- As result, we obtain in our case:

$$\begin{aligned} \dot{k} &= \underbrace{\dot{k}^*}_{(=0)} + [f'(k^*) - (\delta + x)](k - k^*) - (g - g^*) \qquad (16) \\ \dot{g} &= \underbrace{\dot{g}^*}_{(=0)} + (1/\theta) f''(k^*) g^* (k - k^*) \\ &+ (1/\theta)[f'(k^*) - \delta - \rho - \theta x](g - g^*). \qquad (17) \end{aligned}$$

- We arrive at a *two dimensional* dynamic system in *deviation form* denoted by:

$$\dot{\mathbf{Z}} = \mathbf{J}\mathbf{Z}. \tag{18}$$

- According to this, we have:

$$\begin{pmatrix} \dot{k} - \dot{k}^* \\ \dot{g} - \dot{g}^* \end{pmatrix} = \begin{pmatrix} f'(k^*) - \delta - x & -1 \\ f''(k^*)g^*/\theta & (1/\theta)[f'(k^*) - \delta - \rho - \theta x] \end{pmatrix} \begin{pmatrix} k - k^* \\ g - g^* \end{pmatrix}. \tag{19}$$

- Since $\mathbf{J}$ denotes the **Jacobian** matrix (i.e. the matrix of first derivatives evaluated at the steady state), we substitute out $f'(k^*) - \delta$ by the remainder of the expression for the g-isocline given by equation (13). This yields:

$$\begin{pmatrix} \dot{k} - \dot{k}^* \\ \dot{g} - \dot{g}^* \end{pmatrix} = \begin{pmatrix} \rho - (1-\theta)x & -1 \\ f''(k^*)g^*/\theta & 0 \end{pmatrix} \begin{pmatrix} k - k^* \\ g - g^* \end{pmatrix}. \tag{20}$$

Note on Stability: The Role of the Eigenvalues

- Recall that the capital stock k is a predetermined variable (since the initial value of $k(0)$ is known) while consumption g is a jump variable.

- Since we foremost interested in a *stable* convergent adjustment path towards the steady state point, for which the optimal market equilibrium (k^{*}, g^{*}) holds, the dynamic system must exhibit so-called **saddle point** stability.

- It follows that we have to determine $g(0)$ such that under consideration of $k(0)$ we follow a specific *trajectory* towards (k^{*}, g^{*}). This means if the system is not in steady state, $g(0)$ and $k(0)$ must lie on this trajectory. The latter is known as the **saddle path**.

- As we turn to the general solution of the dynamic system later on, we have to consider the corresponding Eigenvectors and Eigenvalues.

- For saddle point stability to hold, it has to be true that the *number of jump variables equals the number of unstable Eigenvalues*. In macroeconomics, this rule is well known as the Blanchard-Kahn condition.

- As we consider a two dimensional system in *continuous* time, we should have **exactly** one negative and positive Eigenvalue, respectively.

- Let us denote the (real and distinct) Eigenvalue by λ. It must be the case that $\lambda_1 > 0$ and $\lambda_2 < 0$ (or vice versa) holds. Note that the Eigenvalues must be of opposite sign and none of them can be equal to zero!

- Note also that (in general) the following rule applies:

$$\mathbf{J} \in \mathbb{R}^{n \times n} \to \text{DET}(\mathbf{J}) = \prod_{i=1}^{n} \lambda_i$$

where DET($\mathbf{J}$) denotes the *determinant* of the Jacobian matrix.

- According to (20) we have:

$$\begin{aligned} \text{DET}(\mathbf{J}) &= \mathbf{J}_{11} \cdot \mathbf{J}_{22} - \mathbf{J}_{21} \cdot \mathbf{J}_{12} \\ &= [\rho - (1-\theta)x] \cdot 0 - [f''(k^*)g^*/\theta] \cdot (-1) \\ &= \underbrace{f''(k^*)}_{<0} \underbrace{g^*}_{\gtreqless 0} / \underbrace{\theta}_{>0} . \end{aligned} \tag{21}$$

- Obviously, $\text{DET}(\mathbf{J}) = \lambda_1 \cdot \lambda_2 < 0$ must hold since this implies $\lambda_1 > 0$ and $\lambda_2 < 0$ (or vice versa).

- For this $g^* > 0$ is strictly required. Therefore, we have to find the corresponding stability condition.

- As we will see, if this condition is fulfilled, the TVC holds automatically — which is required to obtain an *unique* optimal market equilibrium.

Note on Stability: Link between Stability Condition & TVC

- Given that the *capital share* α is defined as

$$0 < [f'(k)k]/f(k) < 1$$

  the following must hold:

$$f(k) > f'(k)k > 0.$$

- As we have $g^* = \underline{f(k^*)} - (\delta + x)k^*$ according to equation (11), it follows:

$$g^* > \underline{f'(k^*)k^*} - (\delta + x)k^* = [f'(k^*) - (\delta + x)]k^*. \qquad (22)$$

- Proof:

$$\begin{aligned} f(k^*) - (\delta + x)k^* &> f'(k^*)k^* - (\delta + x)k^* \\ \rightarrow f(k^*) &> f'(k^*)k^*. \end{aligned}$$

q.e.d.

- In order to strictly ensure $g^* > 0$ to hold, $f'(k^*) > (\delta + x)$ is required.

- Under consideration of equation (13) we get:

$$\rho > (1 - \theta)x. \tag{23}$$

- This is always true for $\theta > 1$, which is empirically most likely the case. Even for $\theta < 1$ it is most plausibly true; e.g. it is also always true for $\rho > x$ because θ is positive.

- The inequality (23) is the *condition for saddle point stability* in the RCK model: given $g^* > 0$ this implies DET($\mathbf{J}$) < 0 according to equation (21).

- The stability condition is necessary and sufficient in order to ensure that the pair of steady state levels (k^*, g^*) characterizes an *unique* market equilibrium which is Pareto optimal.

- To see this consider the following to be true in the steady state (cf. equation (8)):

$$\widehat{g}\Big|_{k^*} = 0 \rightarrow f'(k^*) - \delta = \rho + \theta x. \tag{24}$$

- Given $\iota^* = f'(k^*) - \delta$ and subtracting x from both sides we obtain

$$\iota^* - x = \rho - (1 - \theta)x. \tag{25}$$

- Let us pretend that the interest rate at all times (i.e. also in the steady state) equals the average interest rate which gives $\iota = \iota^* = \bar{\iota}$.

- Hence, $\rho > (1 - \theta)x$ implies $\iota > x$.

- Now let us rewrite the TVC given by equation (9) as follows:

$$\lim_{t\to\infty} D(t)E(t)k(t) = \lim_{t\to\infty} \underbrace{\exp(-\iota \cdot t)}_{D(t)=D(0)\exp(-\iota\cdot t)} \cdot \underbrace{\exp(x \cdot t)}_{E(t)=E(0)\exp(x\cdot t)} \cdot k(t) = 0 \qquad (26)$$

  with $D(0) = E(0) = 1$.

- For $\iota > x$ we indeed observe that

$$\lim_{t\to\infty} \exp[(x-\iota)\cdot t] \cdot k(t) = 0 \qquad (27)$$

  holds. Therefore, the TVC is fulfilled under saddle point stability.

- Note that in this case the household be neither a net lender nor borrower, respectively, at judgment day T.

- This implies *no reallocations* in the steady state which marks it as a stable and unique market equilibrium. The outcome is therefore Pareto optimal.

- To go a step further, the steady state capital stock lies to the left of the *golden rule* one, i.e. $k^* < k_g$ holds.

- Recall that we know from the equations (12) and (13) that $f'(k_g) = \delta + x$ and $f'(k^*) = \delta + \rho + \theta x$ must be considered.

- It follows for $\rho > (1 - \theta)x$ that we observe $f'(k^*) > f'(k_g)$ which then implies $k^* < k_g$ given the production function is strictly concave.

Solution Technique

- Recall that the linearized system in deviation form is given by

$$\dot{\mathbf{Z}} = \mathbf{J}\mathbf{Z}$$

  with the Jacobian matrix $\mathbf{J}$ being defined in equation (20).

- The *general solution* of the system is as follows:

$$\mathbf{Z}^{(*)} = \begin{pmatrix} k - k^* \\ g - g^* \end{pmatrix} = A_1 \cdot \mathbf{h_1} \cdot \text{exp}(\lambda_1 \cdot t) + A_2 \cdot \mathbf{h_2} \cdot \text{exp}(\lambda_2 \cdot t). \qquad (28)$$

- Note that this general solution is valid since it solves the underlying *Eigenvalue problem* given by $\mathbf{J} \cdot \mathbf{h_i} = \lambda_i \cdot \mathbf{h_i}$ for $i = \{1, 2\}$.

- A_1 and A_2 are (arbitrary) constants. $\mathbf{h_1}$ and $\mathbf{h_2}$ are the corresponding Eigenvectors to the Eigenvalues λ_1 and λ_2.

- Since the Eigenvalues remain unaltered through a linear transformation of the system, they can be normalized such that the second entries are equal to one:

$$\begin{aligned} \mathbf{h_1} &= \begin{pmatrix} h'_{11} \\ h'_{21} \end{pmatrix} = \begin{pmatrix} h_{11} \\ 1 \end{pmatrix}, \\ \mathbf{h_2} &= \begin{pmatrix} h'_{12} \\ h'_{22} \end{pmatrix} = \begin{pmatrix} h_{12} \\ 1 \end{pmatrix} \end{aligned}$$

  with $h_{1j} = h'_{1j}/h'_{2j}$ for $j = \{1,2\}$. Without loss of generality, let us assume that $h_{11} < 0$ and $h_{12} > 0$ hold.

- Note that for saddle point stability $\text{DET}(\mathbf{J}) = \lambda_1 \cdot \lambda_2 < 0$ is required. Again, without loss of generality, let $\lambda_1 > 0$ and $\lambda_2 < 0$ denote the unstable and stable Eigenvalue, respectively.

- The saddle path is located on the **stable** *arm* (of the saddle). It follows from the general solution (28) for A_1 set to zero:

$$(g - g^*) = (1/\underbrace{h_{12}}_{>0}) \cdot (k - k^*). \tag{29}$$

- If the initial values $g(0)$ and $k(0)$ are *not* to be found on the saddle path, the system (di)converges asymptotically towards the **un***stable arm* (of the saddle). We arrive at the following expression by setting A_2 equal to zero:

$$(g - g^*) = (1/\underbrace{h_{11}}_{<0}) \cdot (k - k^*). \tag{30}$$

- In order to obtain a convergence along the saddle path, for this kind of system it has to be true *in general* that the constant, which is associated to the unstable Eigenvalue, must be set to zero.

- In our case, this must be true for A_1 since $\lambda_1 > 0$ holds. Otherwise (according to the general solution (28)) the dynamics will not approach zero in the limit.

- If the steady state is a saddle point, the initial consumption level $g(0)$ is uniquely defined.

- Since the capital stock behaves *continuously* over time and its initial value $k(0)$ is known, A_2 can be easily determined.

- Note that for $t = 0$ we have:

$$k(0) - k^* = \underbrace{A_1}_{\overset{!}{=}0} \cdot h_{11} \cdot \underbrace{\exp(\lambda_1 \cdot 0)}_{=1} + A_2 \cdot h_{12} \cdot \underbrace{\exp(\lambda_2 \cdot 0)}_{=1} = A_2 \cdot h_{12} \tag{31}$$

from which it follows $A_2 = (1/h_{12}) \cdot (k(0) - k^*)$.

- From the second equation we then get:

$$g(0) - g^* = A_2 \cdot \underbrace{\exp(\lambda_2 \cdot 0)}_{=1}. \tag{32}$$

- Under consideration of the expression for A_2, the initial level of $g(0)$ is then given by

$$g(0) = g^* + (1/h_{12}) \cdot (k(0) - k^*). \qquad (33)$$

- Not surprisingly, the equations (29) and (33) are identical for $t = 0$.

- Hence, by applying equation (33) it is guaranteed that both $k(0)$ and $g(0)$ lie on the stable arm.

- Finally, the movements of consumption and the capital stock over time ($\forall t > 0$) follow from the general solution (28)) for given constants $A_1(\overset{!}{=} 0)$ and A_2:

$$\begin{aligned} k &= k^* + (k(0) - k^*) \cdot \exp(\lambda_2 \cdot t) && (34) \\ g &= g^* + (1/h_{12})(k(0) - k^*) \cdot \exp(\lambda_2 \cdot t). && (35) \end{aligned}$$

Excursus: Trajectories towards the Unstable Arm

- Suppose that $g(0)$ does *not* lie on the saddle path but instead above or below it.
- In these cases, the system converges asymptotically towards the *un*stable arm along the corresponding trajectories.
- Algebraically, we have to consider *both* constants in the general solution (28)) to be determined:

$$\mathbf{Z} = \tilde{A}_1 \cdot \mathbf{h_1} \cdot \exp(\lambda_1 \cdot t) + \tilde{A}_2 \cdot \mathbf{h_2} \cdot \exp(\lambda_2 \cdot t). \tag{36}$$

- For $k(0)$ given, in $t = 0$ we have:

$$(k(0) - k^*) = \tilde{A}_1 \cdot h_{11} + \tilde{A}_2 \cdot h_{12} \tag{37}$$

$$\rightarrow \tilde{A}_2 = [(k(0) - k^*) - (\tilde{A}_1 \cdot h_{11})]/h_{12}. \tag{38}$$

- The second equation in $t = 0$ yields:

$$\begin{aligned} (\tilde{g}(0) - g^*) &= \tilde{A}_1 + \tilde{A}_2 && (39) \\ \rightarrow \tilde{A}_2 &= (\tilde{g}(0) - g^*) - \tilde{A}_1 && (40) \end{aligned}$$

where $\tilde{g}(0)$ denotes the *known* (e.g. freely set) initial value of consumption which is unequal to $g(0)$, i.e. $\tilde{g}(0) \gtrless g(0)$ holds.

- Plugging (40) into (37) results in

$$\begin{aligned} (k(0) - k^*) &= \tilde{A}_1 \cdot h_{11} + [(\tilde{g}(0) - g^*) - \tilde{A}_1] \cdot h_{12} && (41) \\ \rightarrow \tilde{A}_1 &= [(k(0) - k^*) - (\tilde{g}(0) - g^*) \cdot h_{12}]/(h_{11} - h_{12}). && (42) \end{aligned}$$

- Note that for $\tilde{A}_1 = 0$ to hold, equation (33) must be applied for $\tilde{g}(0) = g(0)$.

- The corresponding time paths for consumption and the capital stock follow from the general solution (28)) for given constants $\tilde{A}_1$ and $\tilde{A}_2$.

Analysis of Local Dynamics in the Phase Diagram

- How about the trajectories to be found in the phase diagram starting at some $k(0) < k^*$?

- Obviously, $g(0)$ must lie below the k-isocline. Otherwise $\widehat{k}$ would be negative leading to a collapse with $k(0)$ at a *finite* time $\tilde{T}$.

- Hence, to have $\widehat{k} > 0$ the following must apply:

$$f[k(0)] - (\delta + x)k(0) > g(0) \tag{43}$$

  (cf. equation (7)). Equivalently, the opposite must hold for $k^* < k(0)$.

- The phase diagrams on the following slides show outcomes where the parameters are specified as follows: $\rho = 0.02$, $\delta = 0.05$, $x = 0.02$, $\theta = 1/0.6$, $a = 1$ and $\alpha = 0.3$. Note that we consider the CD production function in implicit form given by $f(k) = ak^{\alpha}$. We set $k(0)$ equal to 2.

- The steady state levels of k and g are determined as follows (for CD):

$$\dot{g} = 0 \rightarrow f'(k^*) = \alpha a k^{*(\alpha-1)} = \rho + \delta + \theta x \rightarrow k^* = \left(\frac{\rho + \delta + \theta x}{a\alpha} \right)^{\frac{1}{\alpha-1}} > 0 \quad (44)$$

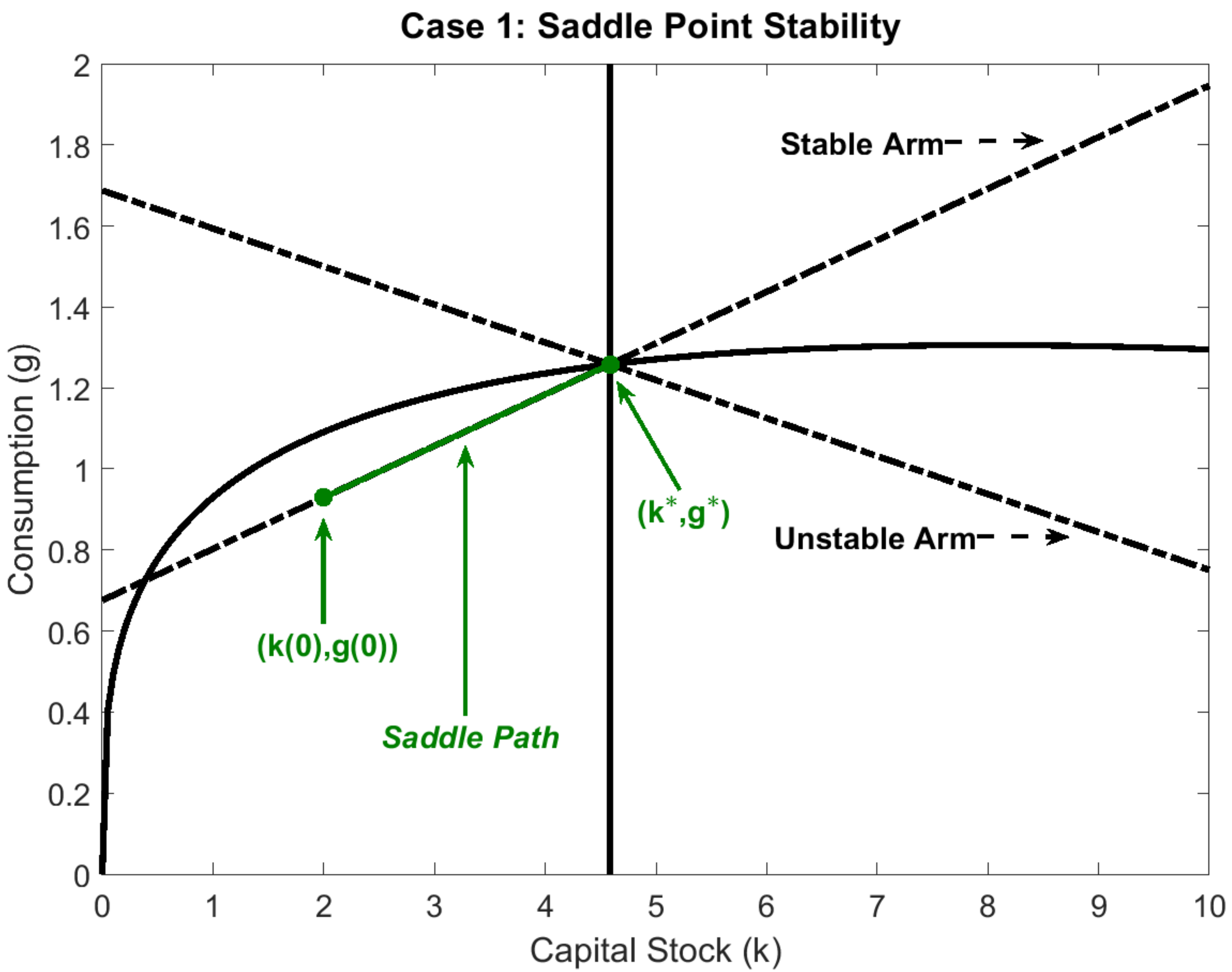
and

$$\dot{k} = 0 \rightarrow g^* = a k^{*\alpha} - (\delta + x) k^* > 0 \quad (45)$$

(cf. equations (14) and (15)). Note that $g^* > 0$ holds given that the condition for saddle point stability (23) applies.

- As we focus on $k(0) < k^*$, **three possible cases** occur.

- In the *first* case, the initial values $k(0)$ and $g(0)$ lie on the stable arm of the saddle. Hence we observe a convergent adjustment towards the steady state (k^*, g^*) along the saddle path. Let us denote the latter by $\tilde{g}(k)$. Therefore, in this case $g(0) \stackrel{!}{=} \tilde{g}[k(0)]$ holds.


Case 1: Saddle Point Stability
Stable Arm
Unstable Arm
(k*,g*)
(k(0),g(0))
Saddle Path
Consumption (g)
Capital Stock (k)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
9
10

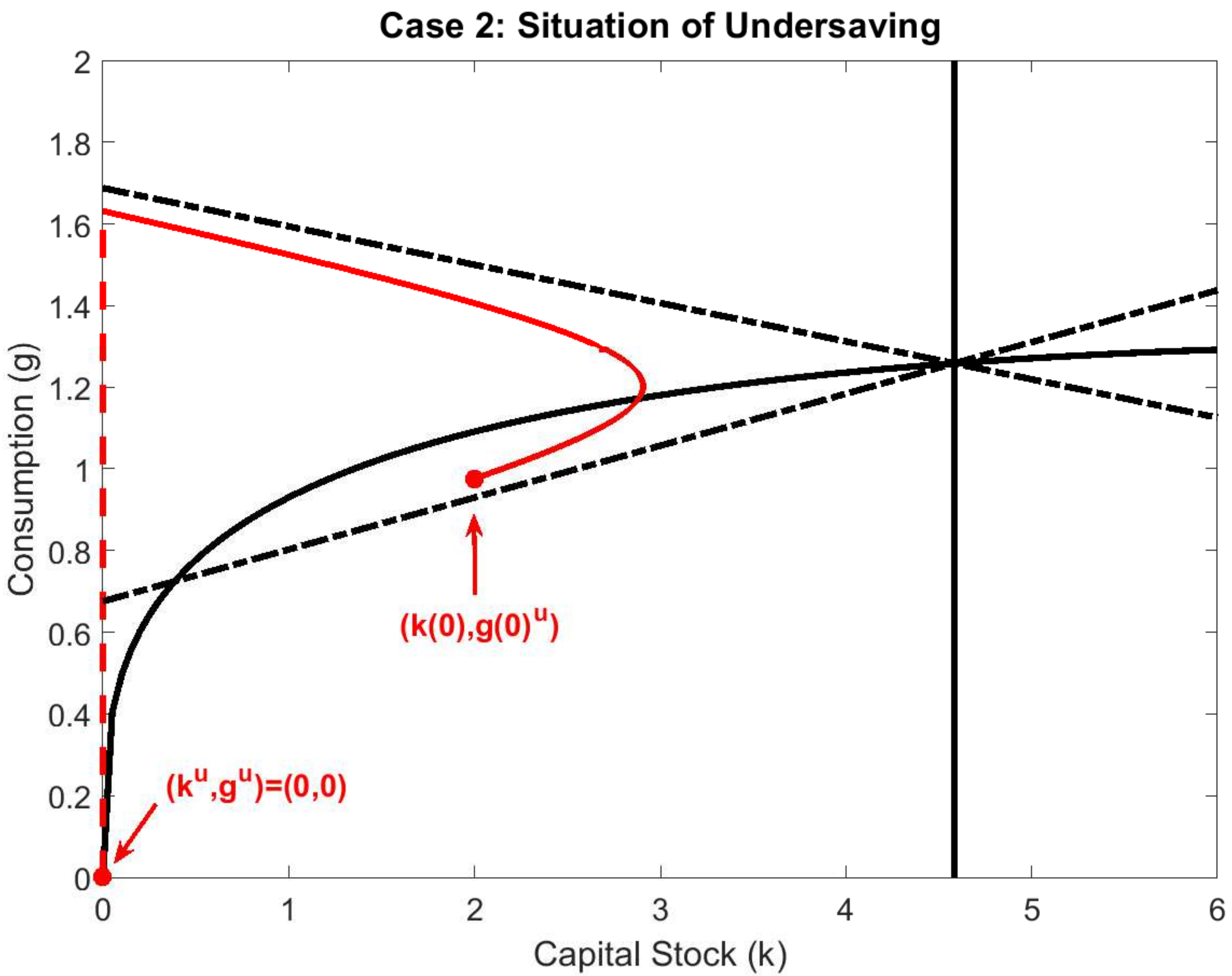
- In the *second* case, for $g(0) \overset{!}{=} g(0)^u > \tilde{g}[k(0)]$ the trajectory must vertically cut through the k-isocline left of the steady state, leading to a collapse in finite time $\tilde{T}$. As a result we obtain $(k^u, g^u) = (0, 0)$.

- From the phase diagram we know that for $g(0) > \tilde{g}[k(0)]$, $\widehat{g} > 0$ holds while we observe $\widehat{k} < 0$ as soon as the corresponding trajectory intersects with the k-isocline.

- It follows that the capital stock declines over time until $k(\tilde{T}) = 0 < k(0)$ is reached. Equation (14) then implies $g(\tilde{T}) = 0$ for $f[k(\tilde{T})] = f(0) = 0$. Hence, without output being produced, nothing can be consumed.

- This specific steady state cannot be optimal since it violates household's first-order condition (3), i.e. the KRR. Since reallocations still apply if we are not in (k^*, g^*), $\widehat{c} > 0$ must hold. However, the latter is not possible given $k(\tilde{T}) = 0$ as described above.

- Economically speaking, the adjustment over time resembles a situation of **undersaving**, i.e. the economy is under capitalized. Since initial saving is too low, $k(\tilde{T}) = 0$ and $f(0) = 0$ lead to g immediately jumps to zero in this case.


Case 2: Situation of Undersaving
Consumption (g)
Capital Stock (k)
(k(0),g(0)^u)
(k^u,g^u)=(0,0)
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6

- In the *third* case, for $g(0) \stackrel{!}{=} g(0)^o < \tilde{g}[k(0)]$, we observe the situation of **oversaving**, i.e. initial saving is too high.

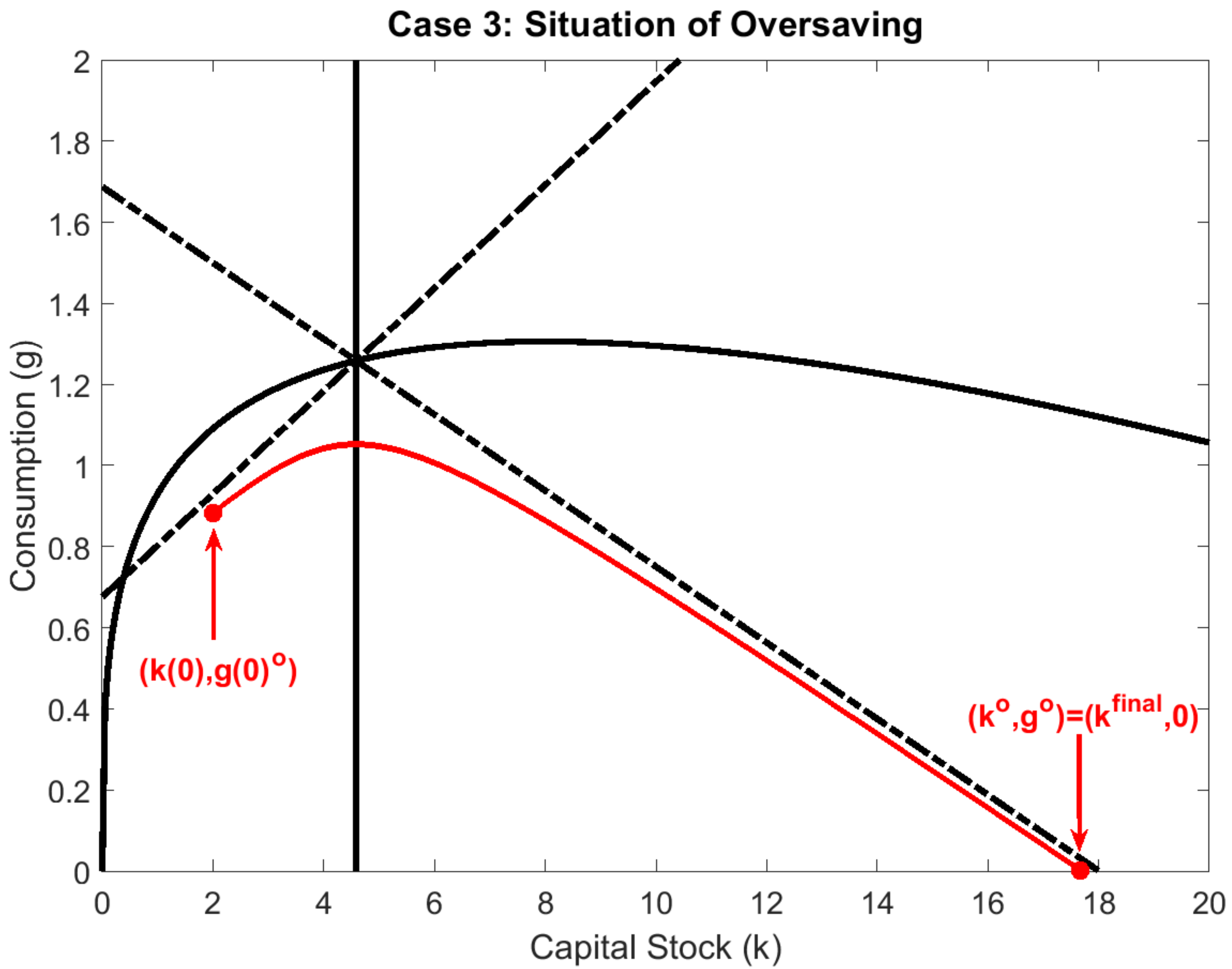
- This means that the household has no incentive to save at a level required to reach g^*.

- As a result the household will not "scarifies" consumption today in favor for consumption tomorrow and, hence, becomes a net lender.

- According to the phase diagram the economy moves towards the *dynamically inefficient region* and ends up at $(k^o, g^o) = (k^{\text{max}}, 0)$.

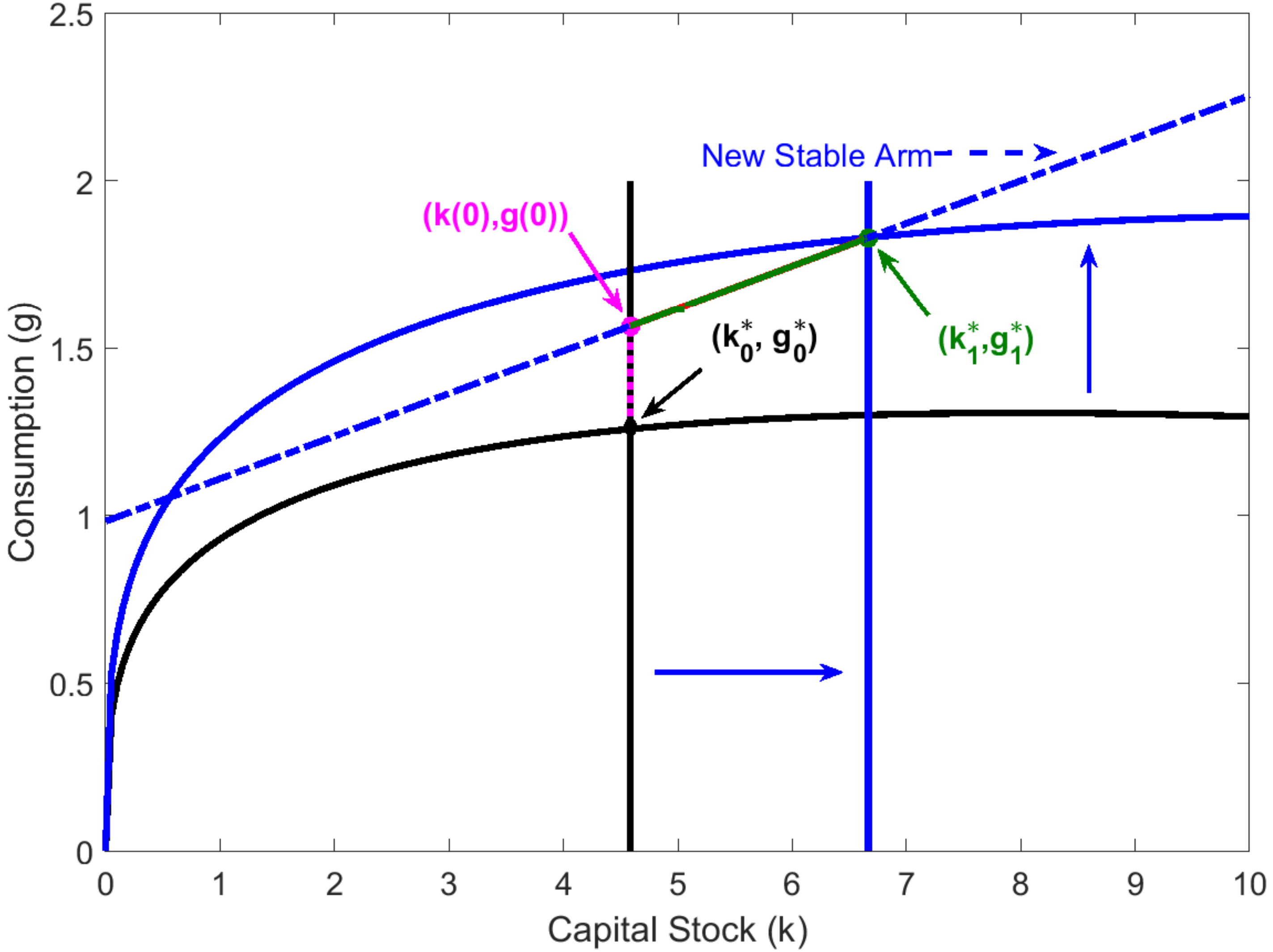
- Note that in the *linear* case we observe that the final steady state is located, in fact, at $(k^o, g^o) = (k^{\text{final}}, 0)$. This must hold since the corresponding trajectory always converges asymptotically towards the unstable arm.

- We already know that any steady state value of k right to k_g violates the TVC (cf. equation (27)). Hence, this steady state cannot be optimal as well.


Case 3: Situation of Oversaving
Consumption (g)
Capital Stock (k)
(k(0),g(0)°)
(k°,g°)=(k^final,0)
0
2
4
6
8
10
12
14
16
18
20
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8

Impact of a positive productivity shock

- We analyze the outcome of a positive productivity shock, i.e. d$a = a_1 - a_0 > 0$ holds. It follows, ceteris paribus, that $f(k)$ and $f'(k)$ rise.
- With respect to the k-isocline, the level of consumption in the steady state must increase to match now a higher level of output to be produced.
- With respect to the g-isocline, with more opportunities at hand, consumption must increase. Since $f'(k) - \delta$ rises, there is more incentive to save. From this it follows a higher degree of capital accumulation.


New Stable Arm
(k(0),g(0))
(k*0, g*0)
(k*1,g*1)
Consumption (g)
Capital Stock (k)
2.5
2
1.5
1
0.5
0
0
1
2
3
4
5
6
7
8
9
10

- Analytically, we determine the pairs of steady state values (k_0^*, g_0^*) and (k_1^*, g_1^*) with a_0 and a_1, respectively, according to equations (44) and (45).

- The linearized dynamic system is now described as follows:

$$\begin{pmatrix} k - k_1^* \\ g - g_1^* \end{pmatrix} = A_2 \cdot \mathbf{h_2} \cdot \text{exp}(\lambda_2 \cdot t) \tag{46}$$

  where we account explicitly for (k_1^*, g_1^*). Again, A_1 must be set to zero since $\lambda_1 > 0$ holds.

- Note that the Jacobian has to be evaluated around the *new* steady state (k_1^*, g_1^*) since we move along the associated stable arm, i.e. on the saddle path.

- Since the exogenous expression for productivity a (besides x, δ, ρ, θ and α) can be found as part of the Jacobian, the latter changes quantitatively for any change in a.

- In this special case, the corresponding entry of interest in the Jacobian becomes:

$$J_{21} = f''\left(k^*\Big|_{a_1}\right) \cdot \frac{g^*\Big|_{a_1}}{\theta}. \tag{47}$$

- The time paths for k and g are now given by

$$k = k_1^* + (k(0) - k_1^*) \cdot \exp(\lambda_2 \cdot t) \tag{48}$$

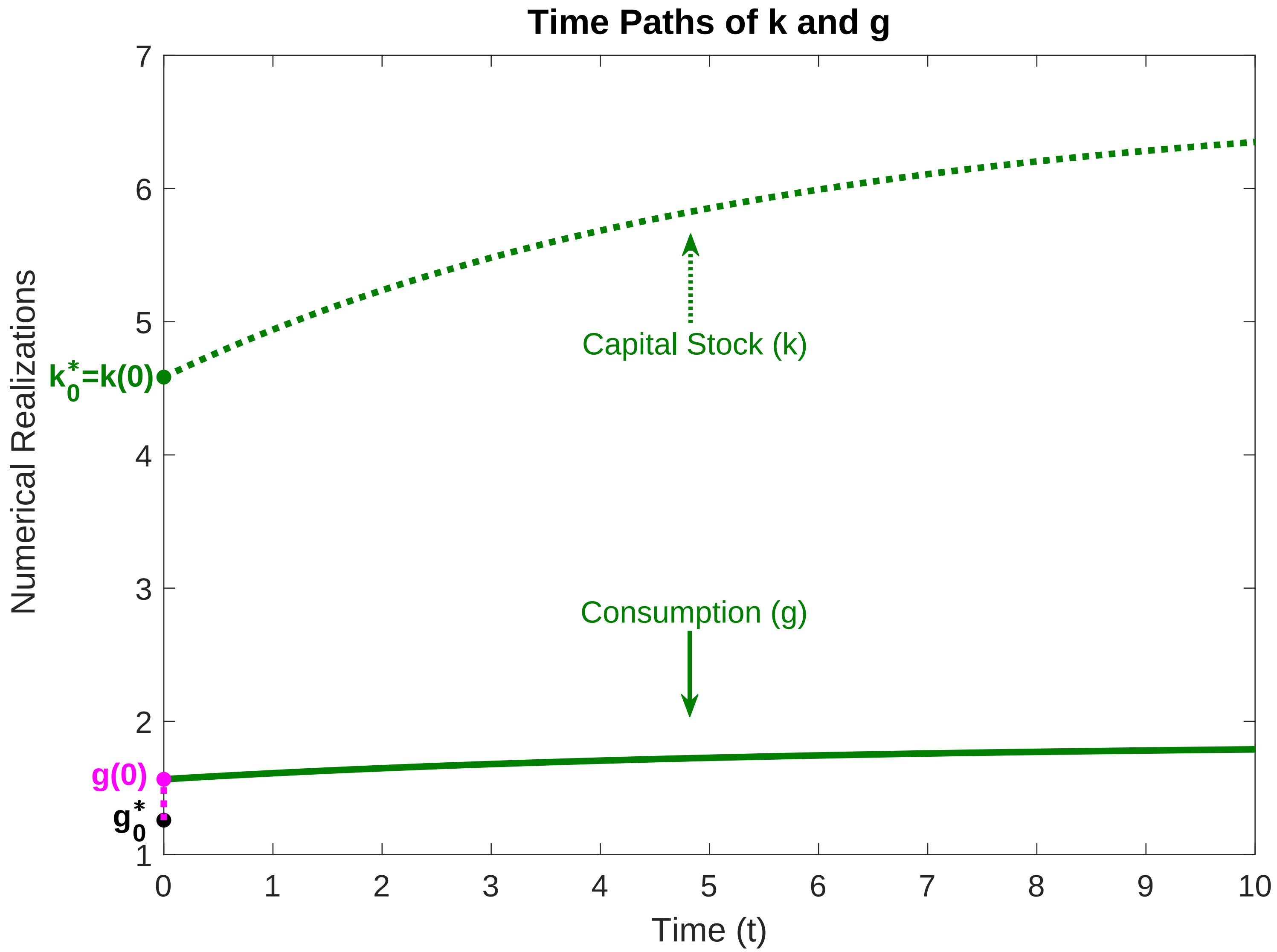
$$g = g_1^* + (1/h_{12})(k(0) - k_1^*) \cdot \exp(\lambda_2 \cdot t) \tag{49}$$

(see equation (46) together with (34) and (35)). Note that $k(0) = k_0^*$ holds.

- Finally, since consumption behaves discontinuously **on impact**, as the shock occurs the variable *jumps* from g_0^* towards

$$g(0) = g_1^* + (1/h_{12}) \cdot (k(0) - k_1^*) \tag{50}$$

(cf. equation (33)), i.e. along the “old” g-isocline. From there on we move directly on the saddle path towards the new steady state (k_1^*, g_1^*).


Time Paths of k and g
Numerical Realizations
Time (t)
Capital Stock (k)
Consumption (g)
k*0=k(0)
g(0)
g*0
7
6
5
4
3
2
1
0
1
2
3
4
5
6
7
8
9
10

References

[1] D. Acemoglu. *Introduction to Modern Economic Growth*. Princeton University Press, Princeton, 2009.

[2] R. Barro and X. Sala-i-Martin. *Economic Growth*. 2nd edition, McGraw-Hill, New York, 2004.

[3] G. Gandolfo. *Economic Dynamics*. Springer, Berlin, Heidelberg, 2010.